3. A Brief Review of Dynamics

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3.1 Overview

This chapter reviews the fundamental governing equations for atmosphere and ocean dynamics. This material lays the groundwork for our later examination of coupled atmosphere–ocean variability. The equations of motion are derived first for a Cartesian coordinate system, and then for a rotating spherical coordinate system. The scales of the terms of the momentum equations are analyzed, and a number of simplifications are developed for later use (including the f-plane and β-plane coordinate systems, the hydrostatic approximation for the atmosphere and the Boussinesq approximation for the ocean, and geostrophic balance and the thermal wind).

3.2 The Equations of Motion

The atmosphere and ocean are both fluids, and are therefore subject to fluid dynamics. The laws of fluid dynamics are identical to those of classical solid mechanics (i.e., Newton's laws and the laws of thermodynamics), but their application differs in some fundamental ways.

In classical physics, the dynamic interactions among solid objects are often described according to the position and momentum of the objects themselves. One well-known example is the gravitational interaction between two planets, which can be formulated according to Newton's laws of motion as

$$\frac{d\mathbf{r}_1}{dt} = \mathbf{v}_1 \qquad \frac{d\mathbf{v}_1}{dt} = \frac{Gm_2}{(\mathbf{r}_1 - \mathbf{r}_2)^2}\hat{\mathbf{r}}$$

$$\frac{d\mathbf{r}_2}{dt} = \mathbf{v}_2 \qquad \frac{d\mathbf{v}_2}{dt} = -\frac{Gm_1}{(\mathbf{r}_2 - \mathbf{r}_1)^2}\hat{\mathbf{r}}$$

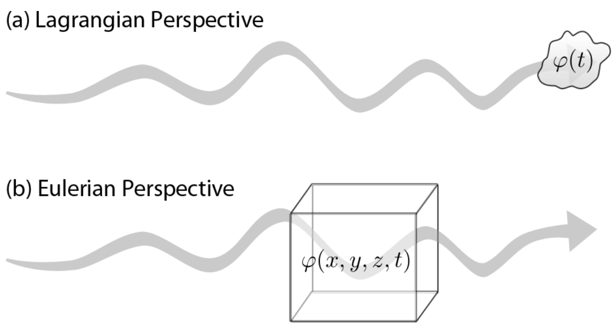

Figure 3.1: Schematic illustration of the differences between the (a) Lagrangian and (b) Eulerian perspectives on fluid dynamics. In the Lagrangian perspective, we track the properties of infinitesimal parcels of the fluid as they are transported by the flow. In the Eulerian perspective, we follow the evolution of the properties of the fluid within a volume fixed in space.

where $\mathbf{r}_i$ and $\mathbf{v}_i$ represent the three-dimensional position and velocity vectors of planet i, m_i is its mass, G is the gravitational constant, and $\hat{\mathbf{r}}$ is a unit vector directed from planet 1 to planet 2.

Unlike planets, which are discrete and easily identifiable, fluids are continuous. Using the same approach to describe fluid dynamics would require us to keep track of the interactions among a very large number of infinitesimal fluid parcels (Fig. 3.1a). This perspective on fluid motion is often called the Lagrangian perspective (after the mathematician Joseph–Louis Lagrange). The Lagrangian approach, while fundamentally correct and useful in some instances, would be very complicated to implement for the entire atmosphere–ocean system. An alternative approach is to examine the evolution of the fluid at fixed points in space from a particular frame of reference (Fig. 3.1b). This perspective on fluid motion is often called the Eulerian perspective (after the mathematician Leonard Euler). Climate models and other descriptions of the Earth system generally use an Eulerian approach, and so we will consider the equations of motion from this perspective.

In the planetary motion example considered above, the state of each planet can be described at any given time according to its position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$. For an Eulerian volume in the atmosphere or ocean, the position $\mathbf{r} = (x, y, z)$ is fixed in time. The state of the fluid in this volume can be described by the velocity $\mathbf{v} = (u, v, w)$, the pressure p, temperature T, density ρ, and any additional components on which these variables depend (such as water vapor q in the atmosphere or salinity S in the ocean). The independent variables are therefore (x, y, z, t), while the dependent variables are $\varphi = (u, v, w, p, \rho, T, q)$ in the atmosphere and $\varphi = (u, v, w, p, \rho, T, S)$ in the ocean. The time evolution of the fluid in the volume can therefore be expressed in the general form as

$$\frac{\partial}{\partial t}\varphi(x, y, z, t) = \mathscr{F}(\varphi, x, y, z, t)$$

where $\mathscr{F}$ is a system of equations determined from fundamental physical laws. Note that

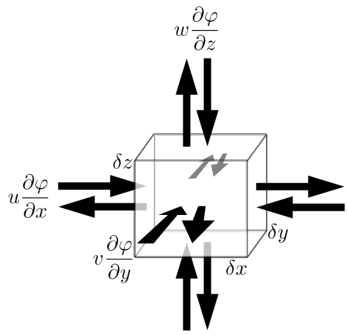

Figure 3.2: Schematic illustration of advection of a fluid property ϕ through the Eulerian volume by the three-dimensional fluid velocity field.

the system $\mathscr{F}$ must contain at least seven equations to solve for seven unknown variables. In practice, the properties of the atmosphere and ocean often allow approximations for one or more of the dependent variables that reduce the order and/or complexity of this system of equations.

3.2.1 The momentum equations

Newton's second law of motion states that force is equal to mass times acceleration:

$$\mathbf{F} = m\mathbf{a} \tag{3.1}$$

where acceleration is the time derivative of velocity ($\mathbf{a} = \frac{d\mathbf{v}}{dt}$) and velocity is the time derivative of the three-dimensional position vector ($\mathbf{v} = \frac{d\mathbf{r}}{dt}$). We can rewrite equation 3.1 as

$$\frac{d\mathbf{v}}{dt} = \frac{1}{m}\mathbf{F} = \mathbf{F}' \tag{3.2}$$

where $\mathbf{F}'$ is the force per unit mass. Recall that Newton's second law is expressed in the Lagrangian framework, so that the velocity $\mathbf{v}$ varies with both position and time:

$$\begin{aligned}\frac{d}{dt}\mathbf{v}(x,y,z,t) &= \frac{\partial\mathbf{v}}{\partial t} + \frac{\partial\mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial\mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial\mathbf{v}}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial\mathbf{v}}{\partial t} + u\frac{\partial\mathbf{v}}{\partial x} + v\frac{\partial\mathbf{v}}{\partial y} + w\frac{\partial\mathbf{v}}{\partial z} \\ &= \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\end{aligned} \tag{3.3}$$

The first term on the right hand side of equation 3.3 ($\frac{\partial\mathbf{v}}{\partial t}$) represents the change of velocity with time in the Eulerian volume located at position (x, y, z). The second term ($\mathbf{v}\cdot\nabla\mathbf{v}$) represents the advection of velocity (momentum per unit mass) into the Eulerian volume by the existing velocity field (as shown schematically in Fig. 3.2). Note that $\mathbf{v}\cdot\nabla$ is an operator acting, in this

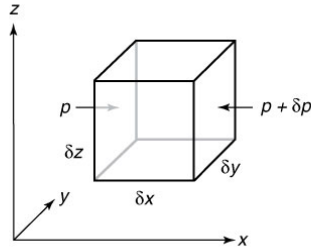

Figure 3.3: Schematic diagram of the pressure gradient force in the x-direction on a cubic Eulerian volume of dimensions $\delta x \times \delta y \times \delta z$. (from oceanworld.tamu.edu).

case, on the velocity field $\mathbf{v}$. This operator also acts on any fluid property φ (e.g., temperature, density, pressure, humidity, or salinity), and relates the Lagrangian time derivative ($\frac{d\varphi}{dt}$) following the fluid motion to the Eulerian derivative ($\frac{\partial\varphi}{\partial t}$) for a fixed volume in space. For the remainder of this text we will write this term as $(\mathbf{v}\cdot\nabla)\varphi$ to specify that $\mathbf{v}\cdot\nabla$ is an operator acting on the fluid property φ.

From equations 3.2 and 3.3, we have the momentum balance for an Eulerian volume:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{F}' \tag{3.4}$$

with $\mathbf{F}'$ the total force per unit mass within the volume. In other words, the velocity of the fluid passing through an Eulerian volume can change due to either momentum advection or outside forces acting on the fluid. The next step is to determine the forces acting on the fluid within the volume. Some of these we have already encountered, such as the gravitational force $\mathbf{g}$ and the vertical pressure gradient force $-\frac{1}{\rho}\frac{\partial p}{\partial z}$ (see equation 1.15). Adding these, the momentum balance becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \mathbf{g} + \mathbf{F}'. \tag{3.5}$$

Pressure in the atmosphere and ocean may vary in the horizontal dimension as well, so that the pressure gradient force is not limited to the vertical coordinate alone. The horizontal pressure gradient force per unit volume can be calculated in the same way as the vertical pressure gradient force (equation 1.14), as the horizontal difference in pressure across a cubic volume:

$$\left(\left[p - \left(\frac{\partial p}{\partial x}\right)\delta x\right]\delta A_{yz} - p\delta A_{yz}\right) + \left(\left[p - \left(\frac{\partial p}{\partial y}\right)\delta y\right]\delta A_{xz} - p\delta A_{xz}\right) = -\left(\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\right)\delta V \tag{3.6}$$

where δV is the volume and the subscripts yz and xz indicate the surface areas in the y–z and x–z planes, respectively (Fig. 3.3). Combining equation 3.6 with equation 1.14 and dividing by $\rho\delta V$, we have an expression for the total pressure gradient force per unit mass:

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}+\frac{\partial p}{\partial z}\right)=-\frac{\nabla p}{\rho}. \tag{3.7}$$

This expression can also be derived for a fluid volume of arbitrary shape (see, e.g., *Vallis*, 2006), and is not dependent on the assumption of a cubic volume used here. Replacing the vertical pressure gradient force in equation 3.5 with the full three-dimensional pressure gradient force (equation 3.7), we have:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{\nabla p}{\rho}+\mathbf{g}+\mathbf{F}' \tag{3.8}$$

where $\mathbf{F}'$ represents additional forces. For the atmosphere and ocean, $\mathbf{F}'$ is dominated by the frictional force $\mathbf{F}_3$, where the subscript 3 indicates that $\mathbf{F}_3=(F_x,F_y,F_z)$ varies in all three dimensions. Note that in practice the friction term also includes momentum dissipation or generation associated with motion at spatial scales that are too small to resolve. For example, many current climate models define the atmosphere or ocean on an Eulerian grid with a grid spacing of $1°\times1°$ (~100 km × 100 km) or more. This grid is too large to resolve many dynamical processes that affect the evolution of the fluid, such as convection (which can occur on horizontal scales less than 10 km). Models often represent the influence of these small-scale motions on the large-scale dynamics via the friction term.

Inclusion of the friction term leads to the vector form of the momentum equation:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{\nabla p}{\rho}+\mathbf{g}+\mathbf{F}_3. \tag{3.9}$$

The first term on the left hand side is the time derivative of velocity (momentum per unit mass) in the Eulerian framework, while the second term is the advection of momentum by the velocity field. The first term on the right hand side is the pressure gradient force (which pushes the fluid from higher pressure to lower pressure both vertically and horizontally), the second term is the gravitational force (which is directed downward in this framework), and the third term is a frictional term as described above. Equation 3.9 is expressed in component form as a system of three equations:

$$\frac{\partial u}{\partial t}+(\mathbf{v}\cdot\nabla)u=-\frac{1}{\rho}\frac{\partial p}{\partial x}+F_x \tag{3.10}$$

$$\frac{\partial v}{\partial t}+(\mathbf{v}\cdot\nabla)v=-\frac{1}{\rho}\frac{\partial p}{\partial y}+F_y \tag{3.11}$$

$$\frac{\partial w}{\partial t}+(\mathbf{v}\cdot\nabla)w=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+F_z \tag{3.12}$$

where we have assumed that gravity acts only in the vertical (z) direction.

3.2.2 The continuity equation

The continuity equation expresses the conservation of mass within the fluid. Consider again an infinitesimal cubic volume of dimensions δx, δy, and δz. The change in the mass of the

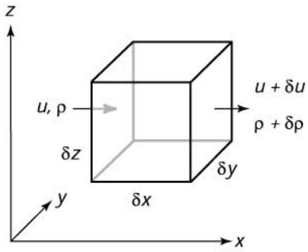

Figure 3.4: Schematic diagram of the fluxes of density in the x-direction into and out of a cubic Eulerian volume of dimensions $\delta x \times \delta y \times \delta z$. (from oceanworld.tamu.edu).

fluid within the volume must be equal to the flux of mass into the volume minus the flux of mass out of the volume (Fig. 3.4). In the x direction, the flow into the volume is expressed by

$$\delta y \delta z \left[(\rho u)(x,y,z) - \left((\rho u)(x,y,z) + \frac{\partial(\rho u)}{\partial x}\delta x \right) \right] = -\frac{\partial(\rho u)}{\partial x}\delta x \delta y \delta z$$

Adding in the y and z directions, the total flow into the volume must be balanced by the change in mass within the volume:

$$\frac{\partial \rho}{\partial t}\delta x \delta y \delta z = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \delta x \delta y \delta z.$$

Cleaning up and combining terms, we obtain the mass continuity equation:

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3.13}$$

As with the pressure gradient force (equation 3.7), equation 3.13 may be derived for a volume of arbitrary shape (see, e.g., *Vallis*, 2006), and is not dependent on the assumption of a cubic volume.

Note that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho + \rho \nabla \cdot \mathbf{v}.$$

This vector identity again allows us to convert between the Lagrangian and Eulerian forms of the mass continuity equation (recall that the Lagrangian time derivative $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho$).

3.2.3 The equation of state and constituent equations

The equation of state expresses the relationship among the various thermodynamic variables (temperature, density, and pressure), and takes the general form $f(p, \rho, T, \mathbf{c}) = 0$ (where $\mathbf{c}$ represents thermodynamically influential constituents, such as water vapor in the atmosphere or salinity in the ocean).

We have already encountered the equation of state for the dry atmosphere in chapter 1 (equation 1.13):

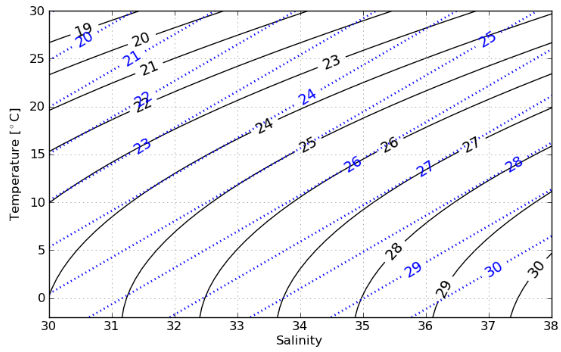

Figure 3.5: Density of seawater near the ocean surface as a function of temperature and salinity according a semi-empirical formula (solid black contours) and a linearized equation of the form given by equations 3.15 and 3.16.

$$p = \rho R_d T$$

This equation may be modified to account for the effects of water vapor phase changes by replacing the temperature T with the virtual temperature T_v:

$$T_v = \frac{T}{1-(e/p)(1-\epsilon)} = (1+0.608q)T \qquad (3.14)$$

where e is the vapor pressure (defined as in equation 2.20), ϵ is the ratio of the molecular weight of water vapor to the mean molecular weight of dry air ($M_v/M_d = 0.622$), and q is the specific humidity (defined as in equation 2.27).

The equation of state for the ocean is more complicated. In its general form, it can be written as

$$\rho = \rho_0 f(T,S,p) \qquad (3.15)$$

where the linearized form of $f(T,S,p)$ is

$$f(T,S,p) = [1-\alpha(T-T_0)+\beta(S-S_0)+\gamma(p-p_0)]. \qquad (3.16)$$

The coefficients α, β, and γ vary with temperature, salinity, and pressure, although near the surface they may be approximated as constant with $\alpha = 2\times10^{-4}\,\mathrm{K}^{-1}$, $\beta = 7.6\times10^{-4}\,\mathrm{ppt}^{-1}$, and $\gamma = 0$. The linearized version of equation 3.15 is not accurate enough for quantitative oceanography (see Fig. 3.5), and should in most instances be replaced by semi-empirical formulae (see, e.g., the seawater python module).

The momentum equations 3.10, 3.11, and 3.12 and the continuity equation 3.13 provide four equations but contain five unknowns (u, v, w, ρ, and p). The equations of state for the atmosphere and ocean provide a fifth equation, but also introduce a sixth and seventh unknown (T and q or S). If the equation of state were such that it did not introduce any new variables, these five equations would be sufficient to describe the evolution of the system. The simplest such case would be that of a constant density fluid, for which the equation of state is simply $\rho = \rho_0$. A fluid for which the density is dependent only on pressure is called a barotropic fluid. In such a fluid, surfaces of constant pressure (isobars) coincide with surfaces of constant density (isopycnals) and surfaces of constant temperature (isotherms), and the flow is tightly constrained. Certain regions of the atmosphere and ocean are approximately barotropic. All other fluids (for which density is dependent on more than just pressure) are called baroclinic.

The concentration of a fluid constituent c (such as water vapor in the atmosphere or salinity in the ocean) can generally be modeled as

$$\frac{\partial c}{\partial t} + (\mathbf{v} \cdot \nabla) c = \frac{1}{\rho}(\Delta c_{\text{src}} - \Delta c_{\text{snk}} + \Delta c_{\text{diff}}) \tag{3.17}$$

where Δc_{src} represents all sources of c in the fluid, Δc_{snk} represents all sinks, and Δc_{diff} represents diffusive mixing. For instance, Δq_{src} for water vapor would include evaporation from the ocean surface, while Δq_{snk} would include condensation to liquid water or ice. For salinity, the sources include evaporation and sea ice formation, while sinks include precipitation and melting of sea ice (note that although these processes do not actually add or remove salt from the ocean, they do change the concentration of salt in the surface layer). An expression like equation 3.17 is a necessary component of the fluid dynamical equations if the constituent c appears in one or more of the other equations. Note that the simplest form of equation 3.17 again results from the assumption that c is constant ($c = c_0$), which is the case for well-mixed constituents.

3.2.4 The thermodynamic equation

The thermodynamic equation is based on the first law of thermodynamics (section 2.2), and is generally expressed as a function of potential temperature (equation 2.5):

$$\frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla)\theta = \frac{\theta}{c_p T}\dot{Q} \tag{3.18}$$

where $\dot{Q}$ is the diabatic heating rate per unit mass, which includes heating due to radiation, latent heating, conduction, and diffusion. Note that the diabatic heating rate $\dot{Q}$ includes radiative heating, which in the atmosphere is dependent on the concentrations of carbon dioxide, ozone, and other gases that absorb either solar or infrared radiation (Fig. 1.10). Inclusion of these concentrations adds further dependent variables, and therefore requires the addition of additional constituent equations in the form of equation 3.17.

3.2.5 Boundary conditions

Together with the equation of state, equations 3.10, 3.11, 3.12, 3.13, 3.17, and 3.18 represent seven equations for seven unknowns. The solution of these equations requires boundary conditions. For example, at the lower boundary of the atmosphere ($z = z_0$), the vertical wind may be assumed to be

$$w = \mathbf{u} \cdot \nabla z_0 = u\frac{\partial z_0}{\partial x} + v\frac{\partial z_0}{\partial y} \tag{3.19}$$

plus some frictional term, where $\mathbf{u} = (u, v)$ is the two-dimensional horizontal velocity vector. This ensures that the surface winds follow the topography (e.g., along mountain slopes), and eliminates any flow of air into or out of the surface. Similar conditions may be imposed at the bottom of the ocean to prevent any flow of water into or out of the ocean floor, as well as at the top of the atmosphere and at the side boundaries of the ocean. At the top of the ocean, pressure is constrained to be equal to the pressure at the bottom of the atmosphere. The vertical velocity at the top of the ocean may be set equal to zero, in which case the top of the ocean acts like a lid. Alternatively, the vertical velocity may be set equal to some condition based on surface pressure, allowing waves, tides, and other variations in sea surface height. Finally, the frictional stresses at the top of the ocean and bottom of the atmosphere must be equal, so that momentum is neither generated nor removed, but is instead conserved and transferred between the two fluids.

3.3 The Effects of Rotation

Section 3.2 provides the fundamental fluid dynamical equations for a fluid in a cartesian coordinate system (e.g., an ocean in a box). In reality, the ocean and atmosphere are located on the surface of Earth, which is approximately spherical. Moreover, the equations discussed in section 3.2 are appropriate only in an inertial frame of reference (i.e., a frame of reference that is fixed relative to the sun and other distant stars). We want to take the position (x, y, z) as fixed with respect to the Earth's surface, which is rotating rapidly relative to the position of the sun and other stars. We must therefore modify the fundamental equations to account for Earth's spherical geometry and rapid rotation.

Figure 3.6 shows a schematic diagram of the spherical coordinate system. This coordinate system is defined relative to the origin located at the center of the Earth, with the position x replaced by the angle λ (longitude), y replaced by the angle ϑ (latitude), and z replaced by the radial vector r. The (x, y, z) geometric coordinates are nonetheless often used in the spherical coordinate system by applying the approximate relations

$$\begin{aligned}
x &= r\cos\vartheta\lambda & \qquad u &= \frac{dx}{dt} = r\cos\vartheta\frac{d\lambda}{dt} \\
y &= r\vartheta & \qquad v &= \frac{dy}{dt} = r\frac{d\vartheta}{dt} \\
z &= r - a & \qquad w &= \frac{dz}{dt}
\end{aligned}$$

where a is the average radius of the Earth ($a = 6.37 \times 10^6$ m). The atmosphere and ocean can be considered as shallow fluids, for which the radius of the Earth is much greater than the depth of the fluid ($a \gg z$). We therefore often replace the spherical coordinate r with the constant a except in the differentiating argument, where r is instead replaced by z (e.g., $r\cos\vartheta\frac{d\lambda}{dt}$ becomes $a\cos\vartheta\frac{d\lambda}{dt}$, whereas $\frac{\partial p}{\partial r}$ becomes $\frac{\partial p}{\partial z}$).

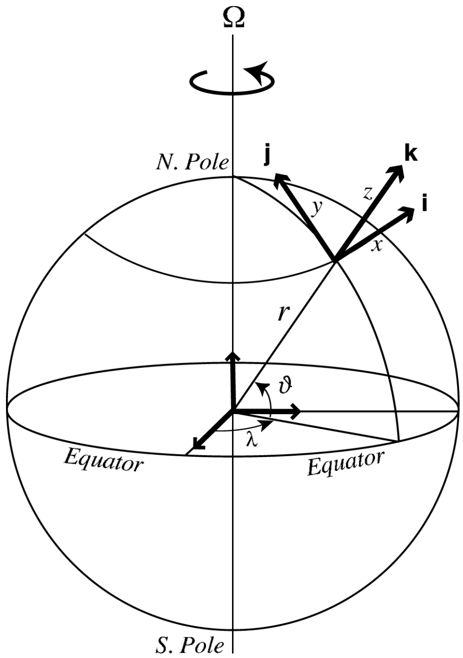

Figure 3.6: Schematic diagram of the spherical coordinate system. The orthogonal unit vectors **i**, **j**, and **k** point in the direction of increasing longitude λ, latitude ϑ, and altitude z. One may apply a quasi-Cartesian coordinate system (x, y, z) as described in the text (from *Vallis*, 2006).

The momentum equations in spherical coordinates are then given by

$$\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u - \left(\frac{u\tan\vartheta}{a}\right)v + \frac{w}{a}u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \tag{3.20}$$

$$\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v + \left(\frac{u\tan\vartheta}{a}\right)u + \frac{w}{a}v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \tag{3.21}$$

$$\frac{\partial w}{\partial t} + (\mathbf{v}\cdot\nabla)w - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \tag{3.22}$$

where the new terms (those involving $1/a$) result from consideration of the curvature of Earth.

The continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial(\rho u)}{\partial x} + \frac{1}{\cos\vartheta}\frac{\partial(\rho v\cos\vartheta)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = 0. \tag{3.23}$$

Note that equation 3.23 is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

when $\nabla\cdot(\rho v)$ is the divergence of the vector $\rho\mathbf{v}$ applied in the true spherical coordinate system (λ, ϑ, r).

The thermodynamic, and constituent equations are effectively unchanged (with the caveat that the definitions of x, y, z, u, v, and w have all been modified to conform to the spherical coordinate system using the relationships listed above). The equation of state has no dependence on the coordinate system.

The rotation of the Earth introduces two apparent forces into the momentum equations, namely the centrifugal force and the Coriolis force. Strictly speaking, neither of these forces are 'real' in the sense of equation 3.1, as we will learn below. For most applications in the atmosphere and ocean the Coriolis force is the more important of the two, but both affect the formulation of the momentum equations in a coordinate system with (x, y, z) fixed relative to the surface of the Earth.

The centrifugal force can be understood in the context of a train that starts to go around a curve in the tracks. A rider on this train feels as though she is being pushed outward toward the side of the car. The outward push that the rider feels is commonly known as the centrifugal force. Newton's first law states that an object will remain at rest or continue moving in a straight line at constant speed unless acted upon by an outside force. Therefore, a force must act upon the rider (and the train) to ensure that she follows the curve (rather than continuing to travel in a straight line). This force is called the centripetal force, which is a real force applied by the rider's seat (or the side wall of the train). The centrifugal force that the rider feels is caused by her tendency to continue going in a straight line while her environment (the train) begins to curve. From the point of view of someone standing beside the tracks, a real force (the centripetal force) is applied to the train. This force causes the train to change directions. From the point of view of the rider, she experiences two forces that cancel each other out: the centrifugal force pushing outward and the centripetal force pushing in. She therefore remains (approximately) stationary within her environment.

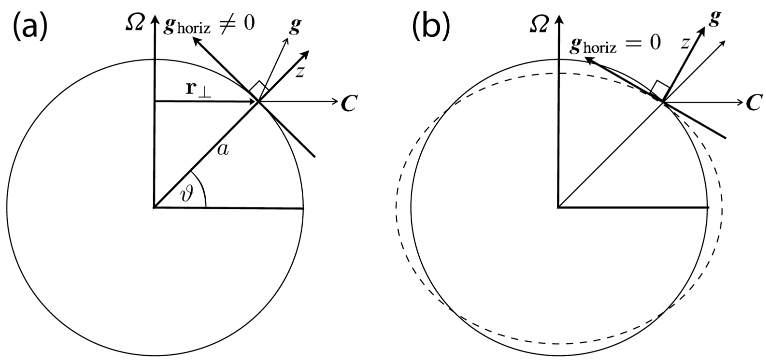

Figure 3.7: (modified from *Vallis*, 2006).

The centrifugal force per unit mass on an object moving in a circle of radius r at an angular velocity Ω is

$$\mathbf{F}_{\rm cen} = \Omega^2 \mathbf{r} \tag{3.24}$$

where $\mathbf{r}$ is a vector of length r from the center of the circle to the position of the object. In a spherical geometry with the axis of rotation through the sphere, this expression becomes

$$\mathbf{F}_{\rm cen} = \Omega^2 \mathbf{r}_\perp \tag{3.25}$$

where $\mathbf{r}_\perp$ is a vector of length $a\cos\vartheta$ (with a the radius of the sphere and ϑ the latitude of the object) in the direction perpendicular to the axis of rotation (as shown in Fig. 3.7a). For Earth, $\Omega = 7.292 \times 10^{-5}$ rad s^{-1} and $a = 6.37 \times 10^6$ m, so that $\mathbf{F}_{\rm cen} = 0.034$ m s^{-2}. The centrifugal force is clearly much smaller than the gravitational force. We can therefore define an effective gravity, which is the sum of the gravitational and centrifugal forces:

$$\mathbf{g} \equiv \mathbf{g}_{\rm eff} = \mathbf{g}_{\rm grv} + \Omega^2 \mathbf{r}_\perp. \tag{3.26}$$

Note, however, that the centrifugal force ($\Omega^2\mathbf{r}_\perp$) is directed perpendicular to the axis of rotation while the true gravitational force ($\mathbf{g}_{\rm grv}$) is directed toward the center of the Earth, so that the effective gravity $\mathbf{g}$ is not perpendicular to the surface of a spherical Earth (Fig. 3.7a). In fact, because the centrifugal effect is largest at the equator and zero at the poles, the radius from the center of the Earth to a point at the equator is approximately 30 km larger than the radius from the center of the Earth to either pole. We can ensure that gravity acts perpendicular to the surface by defining the surface of the Earth according to the geopotential $\Phi = gz$, such that

$$\mathbf{g} = -\nabla\Phi. \tag{3.27}$$

With this adjustment, the effective gravity is oriented perpendicular to the $\Phi = 0$ geopotential surface, which is typically defined as sea level (Fig. 3.7b).

The Coriolis force can be understood in terms of an object launched from the North Pole toward the equator. After the object is launched, it is unaffected by the rotation of the Earth

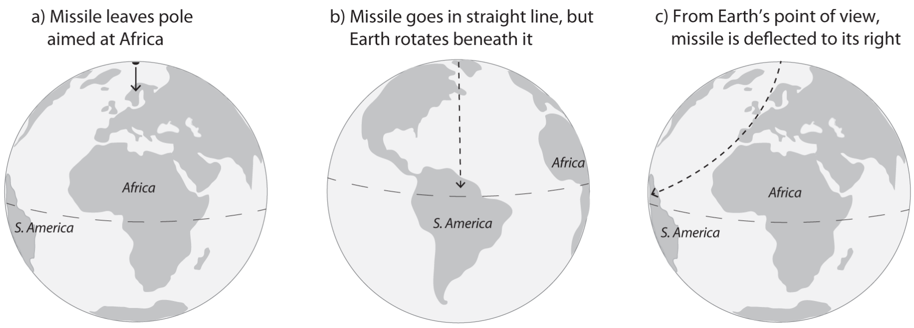

Figure 3.8: Schematic diagram of the flight of a missile launched from the North Pole toward the equator, which illustrates the effect of the Coriolis force. The missile is initially aimed at Africa, but lands in South America instead because Earth rotates beneath it. (from *Vallis*, 2011).

underneath it. By the time it reaches the equator, it will land far to the west of where it was aimed (Fig. 3.8). From an inertial point of view, the object has traveled in a straight line. From the perspective of Earth, the object has been deflected toward the right. If the same object were launched from the equator toward the pole, it would begin its flight with a large eastward velocity and would therefore again appear to be deflected toward the right. The Coriolis force is again not a real force in the Newtonian sense; it is caused by the tendency of an object to go in a straight line while the Earth rotates beneath it. Suppose the object is instead launched in the east–west direction (rather than the north–south direction as before). If the object is launched in the same direction as the Earth's rotation it will move faster than the Earth beneath it, so that the centrifugal force is greater than if the object were stationary on the ground. The object therefore moves away from the axis of rotation (toward the equator). If the object is launched in the opposite direction, the centrifugal force on the object is less than if the object were stationary on the ground and the object will move away from the axis of rotation (toward the pole). No matter which direction the object is launched, it will appear to be deflected toward the right in the northern hemisphere and toward the left in the southern hemisphere.

The Coriolis force per unit mass is

$$\mathbf{F}_{\mathrm{Cor}} = -2\Omega \times \mathbf{v}_{\mathrm{R}} \tag{3.28}$$

where $\mathbf{v}_{\mathrm{R}}$ is the velocity vector relative to the surface of the Earth. The main properties of the Coriolis force can be deduced from equation 3.28: the Coriolis force only acts on bodies that are moving relative to the surface of the Earth (i.e., $|\mathbf{v}_{\mathrm{R}}| \neq 0$), and the Coriolis force acts at right angles to the direction of motion. The three-dimensional components of the vectors Ω and $\mathbf{v}_{\mathrm{R}}$ are

$$\Omega = (0, \Omega\cos\vartheta, \Omega\sin\vartheta)$$
$$\mathbf{v}_{\mathrm{R}} = (u, v, w)$$

so that the vector cross-product is

$$-2\Omega \times \mathbf{v}_\mathrm{R} = (v2\Omega\sin\vartheta - u2\Omega\cos\vartheta, -u2\Omega\sin\vartheta, u2\Omega\cos\vartheta).$$

The full form of the momentum equations in a rotating spherical coordinate system is then

$$\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u - \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)v + \frac{w}{a}u + w\cdot 2\Omega\cos\vartheta = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \tag{3.29}$$

$$\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v + \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)u + \frac{w}{a}v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \tag{3.30}$$

$$\frac{\partial w}{\partial t} + (\mathbf{v}\cdot\nabla)w - \frac{u^2+v^2}{a} - u\cdot 2\Omega\cos\vartheta = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \tag{3.31}$$

where the terms involving Ω are the Coriolis terms. The symbol f is often used to represent the Coriolis term that appears in both horizontal momentum equations ($2\Omega\sin\vartheta$). The equation of state and the continuity, thermodynamic, and constituent equations are not modified by the effects of rotation.

3.4 Simplifications and approximations

We can use our knowledge of the atmosphere and ocean to simplify the full form of the fluid dynamical equations derived in sections 3.2 and 3.3 in a number of ways. For example, both the atmosphere and ocean may be considered shallow fluids, in the sense that they are much broader than they are deep. The atmosphere and ocean are also both stratified, in the sense that lighter fluid is generally located above denser fluid (as shown in chapter 2). This implies that both gravity and buoyancy are important. We may make additional approximations when considering particular types of motion or particular regions of the atmosphere or ocean. Examples of these approximations are described in the remainder of this section.

3.4.1 Scale analysis of the momentum equations

The complete set of fluid dynamical equations derived in sections 3.2 and 3.3 describe all types and scales of motion in the atmosphere and ocean. By focusing on specific types and scales of motion, we can identify which terms in these equations are most important and which terms are relatively unimportant (i.e., small enough to ignore). By eliminating negligible terms from the equations, we can both simplify the equations of motion and filter out other unwanted types and scales of motion. This allows us to better describe the motion of interest.

For example, we can identify the characteristic scales for the horizontal velocity (U), vertical velocity (W), length (L), height (H), and horizontal and vertical pressure fluctuations ($\delta P_{xy}/\rho$ and $\delta P_z/\rho$) for a variety of motions in the atmosphere and ocean. Table 3.1 lists these characteristic scales for two types of motion: synoptic-scale disturbances (i.e., high- or low-pressure systems that extend over ~1000 km) in the mid-latitude atmosphere and the wind-driven circulation in the mid-latitude ocean. For example, synoptic-scale extratropical cyclones or anticyclones extend over regions of approximately 1000 km, and have horizontal velocities on the order of $10\,\mathrm{m\,s^{-1}}$. Note that characteristic scales are generally rounded to the nearest power of 10. Table 3.1 also provides a characteristic scale for the Coriolis term $f = 2\Omega\sin\vartheta$, which in the mid-latitudes is approximately equal to $f_0 = 2\Omega\sin(45°) = 2\Omega\cos(45°) \approx 10^{-4}\,\mathrm{s^{-1}}$. The characteristic time scale for a fluid motion can be calculated as the ratio between the length scale L and the horizontal velocity scale U (T = L/U).

Plugging these characteristic scales into the momentum equations (Tables 3.2 and 3.3) allows us to identify which terms are most important and which terms are relatively unimportant. In both types of motion, the horizontal momentum equation is dominated by the Coriolis term ($u2\Omega\cos\vartheta$ in the x-equation) and the pressure gradient term ($-\frac{1}{\rho}\frac{\partial p}{\partial x}$). Motion for which these two forces are exactly in balance (i.e., the Coriolis force is equal and opposite to the pressure gradient force) is called geostrophic motion, and will be discussed in more detail in section 3.4.4. The vertical momentum equation is clearly dominated by the pressure gradient term and the gravitational term in both cases. Setting the other terms equal to zero retrieves the hydrostatic balance (equation 1.15), in which the pressure gradient force exactly balances the gravitational force. Based on this analysis, motions associated with both synoptic-scale disturbances in the mid-latitude atmosphere and the wind-driven circulation in the mid-latitude ocean are approximately geostrophic and approximately hydrostatic.

Note that we have ignored the frictional term in this scale analysis. The frictional force per unit mass is generally quite small for continuous large-scale motions in both the atmosphere and ocean, so that this term can be comfortably ignored. However, as mentioned in section 3.2.1, the frictional term is often used in models to represent the effects of small-scale motions on the (discrete) large-scale flow. In this case, its form is not immediately obvious, and scale analysis is not possible.

Different types of motion in the atmosphere and ocean have different characteristic scales, so that this scale analysis for large-scale motion in the mid-latitudes cannot be generalized for all situations. However, as mentioned above, the atmosphere and ocean may be considered shallow (H≪L), so that W≪U for large-scale motions in both fluids. Given also that f_0 varies with latitude from 0 to approximately 1.5×10^{-4}, terms involving w/a or $w\cos\vartheta$ can be eliminated from the momentum equations for large-scale motion. When coupled with the assumption of hydrostatic balance this gives the primitive equations:

Table 3.1: Characteristic scales for the variables in the horizontal momentum equations for (**A**) a synoptic-scale disturbance in the mid-latitude atmosphere and (**O**) the wind-driven circulation in the mid-latitude ocean.

	U	W	L	H	$\delta P_{xy}/\rho$	$\delta P_z/\rho$	$f0$	T=L/U
A	$10\,\mathrm{m\,s^{-1}}$	$10^{-2}\,\mathrm{m\,s^{-1}}$	$10^6\,\mathrm{m}$	$10^4\,\mathrm{m}$	$10^3\,\mathrm{m^2\,s^{-2}}$	$10^5\,\mathrm{m^2\,s^{-2}}$	$10^{-4}\,\mathrm{s^{-1}}$	$10^5\,\mathrm{s}$
O	$10^{-1}\,\mathrm{m\,s^{-1}}$	$10^{-5}\,\mathrm{m\,s^{-1}}$	$10^5\,\mathrm{m}$	$10\,\mathrm{m}$	$1\,\mathrm{m^2\,s^{-2}}$	$10^2\,\mathrm{m^2\,s^{-2}}$	$10^{-4}\,\mathrm{s^{-1}}$	$10^6\,\mathrm{s}$

Table 3.2: Scale analysis of the horizontal momentum equations using the characteristic scales listed in Table 3.1.

x-Eq.	$\frac{\partial u}{\partial t}$	$+(\mathbf{v}\cdot\nabla)u$	$-v2\Omega\sin\vartheta$	$-\frac{uv\tan\vartheta}{a}$	$+\frac{uw}{a}$	$+w2\Omega\cos\vartheta$	$=$	$-\frac{1}{\rho}\partial p\partial x$
y-Eq.	$\frac{\partial v}{\partial t}$	$+(\mathbf{v}\cdot\nabla)v$	$+u2\Omega\sin\vartheta$	$+\frac{u^2\tan\vartheta}{a}$	$+\frac{vw}{a}$		$=$	$-\frac{1}{\rho}\partial p\partial y$
Scales	U/T	U^2/L	$f_0\mathrm{U}$	U^2/a	UW/a	$f_0\mathrm{W}$		$^{\delta P_{xy}}/_{\rho}/\mathrm{L})$
A ($\mathrm{m\,s^{-2}}$)	10^{-4}	10^{-4}	10^{-3}	10^{-5}	10^{-8}	10^{-6}		10^{-3}
O ($\mathrm{m\,s^{-2}}$)	10^{-7}	10^{-7}	10^{-5}	10^{-9}	10^{-12}	10^{-8}		10^{-5}

Table 3.3: Scale analysis of the vertical momentum equation using the characteristic scales listed in Table 3.1.

z-Eq.	$\frac{\partial w}{\partial t}$	$+(\mathbf{v}\cdot\nabla)w$	$-\frac{u^2+v^2}{a}$	$-u2\Omega\cos\vartheta$	$=$	$-\frac{1}{\rho}\partial p\partial x$	$-g$
Scales	W/T	UW/L	U^2/a	$f_0\mathrm{U}$		$^{\delta P_z}/_{\rho}/\mathrm{H}$	g
A ($\mathrm{m\,s^{-2}}$)	10^{-7}	10^{-7}	10^{-5}	10^{-3}		10	10
O ($\mathrm{m\,s^{-2}}$)	10^{-9}	10^{-9}	10^{-9}	10^{-5}		10	10

$$\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u - \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \tag{3.32}$$

$$\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v + \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \tag{3.33}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{3.34}$$

The primitive equations are often used in models to simulate large-scale motion in the atmosphere and ocean. In the 1960s, when computers were much less powerful, these equations were the most complex form of the fluid dynamical equations that could be solved numerically by computers in a reasonable amount of time. The formulation of the primitive equations (i.e., the terms that are included) is fundamentally dependent on scale analysis.

3.4.2 The f-plane and the β-plane

If we are interested in the fluid dynamics of a small region of the atmosphere and ocean, we may treat that region as a plane for which the Coriolis term $f = 2\Omega\sin\vartheta$ is constant (i.e., $f = f_0$). In this approximation, we can also ignore the curvature terms (the fluid is defined on a horizontal plane so that $a \to \infty$), so that the momentum equations become

$$\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \tag{3.35}$$

$$\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \tag{3.36}$$

$$\frac{\partial w}{\partial t} + (\mathbf{v}\cdot\nabla)w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \tag{3.37}$$

and the continuity equation is identical in form to equation 3.13 (i.e., the term $\frac{1}{\cos\vartheta}\frac{\partial y}{\partial\vartheta}$ drops out of equation 3.23).

The simpler geometry of the plane may be retained while taking the latitudinal variations of f into account by defining a β-plane, for which f is calculated by linearizing the Coriolis term around a constant f_0:

$$f = f_0 + \beta y.$$

3.4.3 Applying and extending hydrostatic balance

The hydrostatic balance (equation 1.15) holds for large-scale motions in the atmosphere and ocean (section 3.4.1). The fluid dynamical equations for large-scale motions in the atmosphere can therefore be simplified by changing from an (x, y, z) coordinate system to an (x, y, p) coordinate system (i.e., replacing the vertical coordinate z with pressure p). To derive the new equations, let $\varphi'(x, y, z(x, y, p, t), t) = \varphi(x, y, z, t)$, so that

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi'}{\partial x} + \frac{\partial\varphi'}{\partial z}\frac{\partial\varphi}{\partial x}$$
$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi'}{\partial y} + \frac{\partial\varphi'}{\partial z}\frac{\partial\varphi}{\partial y}$$
$$\frac{\partial\varphi}{\partial t} = \frac{\partial\varphi'}{\partial t} + \frac{\partial\varphi'}{\partial z}\frac{\partial\varphi}{\partial t}$$

and, using the hydrostatic balance,

$$\frac{\partial\varphi}{\partial p} = \frac{\partial\varphi'}{\partial z}\frac{\partial z}{\partial p} = -\frac{1}{\rho g}\frac{\partial\varphi'}{\partial z}.$$

Using the geopotential $\Phi = gz$ (equation 3.27) and the equation of state for the atmosphere (equation 1.13), the hydrostatic balance can be expressed as

$$\frac{\partial\Phi}{\partial p} = -\frac{1}{\rho} = -\frac{R_d T}{p}. \tag{3.38}$$

Note that the horizontal pressure gradient in pressure coordinates is by definition zero, so that

$$0 = \nabla_p p = \nabla_z p + \frac{\partial p}{\partial z}\nabla_p z = \nabla_z p - \rho g \nabla_p z$$

where the subscripts p and z indicate the vertical coordinate. The horizontal pressure gradient in z-coordinates is then

$$\frac{1}{\rho}\nabla_z p = g\nabla_p z = \nabla_p \Phi \tag{3.39}$$

In other words, there is no pressure gradient force on a constant pressure surface ($\nabla_p p = 0$), but there is a gravitational force because the constant pressure surface is not perpendicular to gravity (i.e., fluid flows 'downhill').

Converting the momentum equations to pressure coordinates yields

$$\frac{\partial u}{\partial t} + (\mathbf{u}\cdot\nabla_p)u + \omega\frac{\partial u}{\partial p} - fv = -\frac{\partial \Phi}{\partial x} + F_x \tag{3.40}$$

$$\frac{\partial v}{\partial t} + (\mathbf{u}\cdot\nabla_p)v + \omega\frac{\partial v}{\partial p} + fu = -\frac{\partial \Phi}{\partial y} + F_y \tag{3.41}$$

$$\frac{\partial \Phi}{\partial p} = -\frac{R_d T}{p} = -1\rho \tag{3.42}$$

where $\mathbf{u}$ is the two-dimensional velocity vector along pressure surfaces and $\omega = \frac{dp}{dt}$ is the vertical velocity in pressure coordinates. Equation 3.42 results from the assumption of hydrostatic balance (i.e., no acceleration in the vertical direction). Pressure and density are in hydrostatic balance, so that there are no horizontal variations in density on a pressure surface. The continuity equation then becomes

$$\nabla_p \cdot \mathbf{u} + \frac{\partial \omega}{\partial p} = 0. \tag{3.43}$$

This form of the continuity equation is the same as that for a fluid of constant density. The thermodynamic equation is given by the first law of thermodynamics (equation 2.2):

$$\dot{Q} = c_p\frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} = c_p\left(\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla_p)T + \omega\frac{\partial T}{\partial p}\right) - \frac{1}{\rho}\omega. \tag{3.44}$$

Using pressure coordinates simplifies the fluid dynamical equations (particularly the pressure gradient terms and the continuity equation), but complicates the lower boundary conditions. Recall that in (x, y, z) coordinates we generally set $w = \mathbf{u}\cdot\nabla z_0$, so that there is no flow into or out of the surface. Unlike z_0, the surface pressure p_0 changes in both time and space, so that the lower boundary must be determined from scratch at every time step. These approximations are therefore mainly useful in the free atmosphere (i.e., the part of the atmosphere unaffected by the surface). Many atmospheric models, particularly those with complex topography, use the terrain-following vertical coordinate $\sigma = p/p_0$ in place of p.

We can derive a similar set of equations for the ocean, not by replacing z with p but by assuming that the density p is constant everywhere but where it is coupled to gravity. This approximation is useful because variations in density in the ocean are small relative to the mean density. Setting $\rho = \rho_0$ simplifies the horizontal pressure gradients (which are now solely determined by variations in the surface height) and yields a modified continuity equation

$$\nabla \cdot \mathbf{v} = 0. \tag{3.45}$$

Note that the form of equation 3.45 is equivalent to the form of the continuity equation for constant pressure coordinates (equation 3.43).

Even though $\rho - \rho_0$ is small, gravity is large; therefore, variations in density are important where coupled to gravity (i.e., in the vertical momentum equation). Assume that $p = p_0 + p'$, so that

$$\frac{\partial p_0}{\partial z} = -g/rho_0$$

and

$$\frac{\partial p'}{\partial z} = -g(\rho - \rho_0).$$

Define the buoyancy b as

$$b \equiv -g\frac{\rho - \rho_0}{\rho_0}. \tag{3.46}$$

Neglecting friction, the vertical momentum equation then becomes

$$\frac{\partial w}{\partial t} + (\mathbf{v} \cdot \nabla)\, w = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b. \tag{3.47}$$

This approximation is called the Boussinesq approximation after the mathematician Joseph Boussinesq. The Boussinesq approximation is particularly useful for studying buoyancy oscillations in the upper ocean. Moreover, this approximation does not depend on the hydrostatic balance, and can therefore be used to study convection.

3.4.4 Geostrophic balance and the thermal wind

As hinted in section 3.4.1, a geostrophic flow is one for which the horizontal pressure gradient force and the Coriolis force balance. The approximation of geostrophic balance is fundamental to much of meteorology and oceanography, particularly in mid-latitudes. Formal derivations are provided in virtually all textbooks that include ocean or atmosphere dynamics (see, e.g., *Gill*, 1982; *Holton*, 1992; *Vallis*, 2006; *Marshall and Plumb*, 2008).

The evolution of a geostrophic flow proceeds as follows. Suppose that a horizontal pressure gradient develops in the atmosphere or ocean. This pressure gradient generates an acceleration of the flow that is directed from the high pressure region to the low pressure region. As the fluid moves along this gradient, it is deflected by the Coriolis force (to the right in the northern hemisphere, and to the left in the southern hemisphere. In order for the flow to reach a steady state, the pressure gradient force must be exactly balanced by the Coriolis force. In this case, the direction of the flow must be perpendicular to both the pressure gradient force and the Coriolis force. In other words, a geostrophic flow follows contours of constant pressure (isobars), as illustrated in Fig. 3.9. In either hemisphere, the geostrophic flow around low pressure

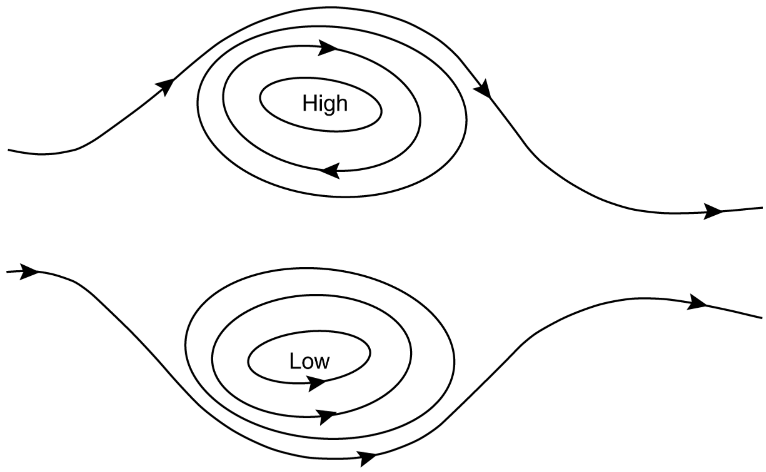

Figure 3.9: Schematic diagram of geostrophic flow in the northern hemisphere. Flow is parallel to lines of constant pressure (isobars), and is oriented counter-clockwise (in the same sense as Ω) around low pressure systems but clockwise around high pressure systems. In the southern hemisphere, flow is oriented clockwise around low pressure systems and counter-clockwise around high pressure systems (from *Vallis*, 2006).

systems is in the same sense as Earth's rotation, and is said to be cyclonic. Geostrophic flow around high pressure systems is in the opposite sense, and is said to be anticyclonic.

We can define the geostrophic velocity according to

$$f u_g = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -\left.\frac{\partial \Phi}{\partial y}\right|_p \tag{3.48}$$

$$-f v_g = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\left.\frac{\partial \Phi}{\partial x}\right|_p \tag{3.49}$$

where the subscript p indicates differentiation on surfaces of constant pressure. This balance is satisfied in regions where friction is small (i.e., away from boundaries) and where the horizontal acceleration is small relative to the coriolis term. From a scaling perspective, this translates into the condition that $\mathrm{U}/\mathrm{L} \ll f_0$, where U is the characteristic velocity scale, L is the characteristic length scale, and f_0 is the characteristic scale of the Coriolis term f. This condition is used to define the Rossby number

$$\mathrm{Ro} \equiv \frac{\mathrm{U}}{f_0 \mathrm{L}} \tag{3.50}$$

such that the geostrophic balance holds for $\mathrm{Ro} \ll 1$.

A sharper pressure gradient can only be balanced by a stronger Coriolis force, so that the geostrophic wind is stronger for sharper pressure gradients. This can be understood by

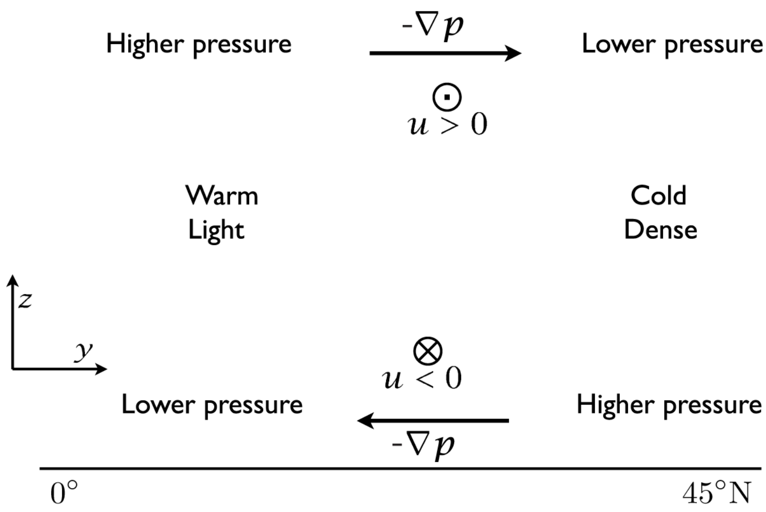

Figure 3.10: Schematic diagram of the thermal wind during northern hemisphere winter. By hydrostatic balance the vertical pressure gradient is greater where the fluid is colder and denser. The pressure gradients form as shown, where ’higher’ and ’lower’ mean relative to the mean at that height. The horizontal pressure gradient forces are balanced by the Coriolis force, which produces westward flow (⊗) at low altitude and eastward flow (⊙) at high altitude (modified from *Vallis*, 2006).

thinking of the spaces between contours of constant pressure (or geopotential height) as channels through which a fixed amount of fluid must pass. A narrower channel then requires a higher velocity to push the fluid through.

If the Coriolis force is constant (such as on an f-plane) and the density does not vary in the horizontal direction then the geostrophic wind is non-divergent, in the sense that the horizontal flow does not lead to fluid mass increasing in certain locations (convergence) while decreasing in others (divergence). However, in the real world friction reduces wind speeds near the surface. The Coriolis force is dependent on the wind speed, and therefore no longer exactly balances the pressure gradient force. The winds are therefore directed slightly toward lower pressures (rather than along isobars, as in the exact geostrophic balance). Mass therefore converges in low pressure systems and diverges from high pressure systems. Conservation of mass requires that convergence be balanced by upward motion (which often leads to condensation of water vapor and precipitation), while divergence must be balanced by downward motion (which is associated with clear, dry conditions). Accordingly, near-surface low pressure systems (cyclones) are often accompanied by cloudy skies and precipitation, while near-surface high pressure systems (anticyclones) are often associated with clear, dry conditions.

Differentiating the constant pressure form of the geostrophic balance with respect to pressure yields

$$f\frac{\partial u_g}{\partial p} = -\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial y} = \frac{R_d}{p}\frac{\partial T}{\partial y} \tag{3.51}$$

where we have used the hydrostatic equation (in the form of equation 3.38). This relation is called the thermal wind equation, and indicates that the vertical shear of the geostrophic wind is a function of the horizontal temperature gradient on constant pressure surfaces. This relationship is illustrated in Fig, 3.10. The pressure gradient near the surface results in an acceleration from the higher pressure in the colder midlatitudes to the lower pressure near the warmer tropics. In geostrophic balance, this results in a westward (easterly) wind near the surface. The warm, light air in the tropical region expands, resulting in higher pressure aloft, while the cold, dense air in mid-latitudes contracts, resulting in lower pressure aloft. This creates a pressure gradient directed from the tropics toward the mid-latitudes, which in geostrophic balances results in an eastward (westerly) wind aloft. This exact situation is observed in the subtropics, where the surface winds (the subtropical trade winds) are westward and the high-altitude winds (the subtropical jet) are eastward.

Horizontal pressure gradients in the ocean, which arise from either variations in the surface height of the ocean or variations in the distribution of density, can be approximated by calculating the ocean dynamic height. Hydrostatic balance, which is an excellent approximation for almost all conditions in the ocean (with the exception of intense convection), can be expressed as

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\alpha$$

where $\alpha = 1/\rho$ is the specific volume. Oceanographers often work with specific volume anomalies relative to a reference specific volume at $S = 35$ ppt (parts per thousand) and $T = 0°$C (i.e., $\alpha = \alpha_{35,0,p} + \delta\alpha$). Integrating equation 3.52 yields an expression for the difference between two geopotential surfaces

$$\Phi_1 - \Phi_2 = \int_{p_1}^{p_2} \alpha\, dp.$$

Now define the ocean dynamic height D as

$$D = \int_{p_1}^{p_2} \delta\alpha\, dp \tag{3.52}$$

so that

$$\nabla_p D = \nabla_p \Phi_1 - \nabla_p \Phi_2. \tag{3.53}$$

If the geopotential surface Φ_2 is flat (i.e., $\nabla_p \Phi_2 = 0$), then D describes the pressure gradient force at level 1. Level 2 is generally defined to be a reference level far enough below the surface that the flow is weak relative to the surface flow (e.g. 1000 dbar, or ∼1000 m below sea level), in which case $\nabla_p \Phi_2 \approx 0$.

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